Multiple Impulse Solutions to McKean's Caricature of the Nerve Equation. II. Stability

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Abstract

We study McKean's caricature of a nerve conduction equation

$$(1)\qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u + H(u-a) + v, \qquad 0 < a < 1,$$

$$\frac{\partial v}{\partial t} = bu - cv, \qquad b < 0,\ c > 0,$$

where H is the Heaviside function. It is proved that an n-ple impulse solution resembling the superposition of n *unstable* solitary impulses has at most $2n-1$, and at least n, unstable modes: exactly n unstable modes corresponding to the amplitudes and the rest of them corresponding to the spacings. The n amplitude modes always exist. ~~We prove also~~ that for an n-ple impulse solution resembling the superposition of n *stable* solitary impulses, there are at most $n-1$ unstable modes and all of them are of spacing type.

Introduction

In the first part of this series [15], we proved that for fixed a, b, c (in a reasonable range) and arbitrary positive integer n, the system (1) has countably many multiple impulse solutions consisting of n widely spaced solitary pulses.

The issue of the stability of multiple impulse solutions is particularly interesting. At some initial time, an impulse is subjected to a perturbation which is bounded spacially. If the system responds by causing the perturbation to grow in time, thereby changing the form of the impulse, the impulse is unstable. If the perturbation decays leaving the impulse traveling with the same form, it is stable.

Stability of solitary impulses of McKean's caricature is clear. Rinzel and Keller [14] demonstrated by computing a growth rate of the instability that the slow solitary impulse is unstable, and conjectured that the fast solitary impulse is stable. Feroe [3] completed Rinzel and Keller's work by computing a winding number. [3] proved numerically that the fast solitary impulse is stable and the slow solitary impulse has exactly one unstable mode.

Evans [1] studied the stability of multiple impulse solutions for the general Hodgkin–Huxley system. He found a complex analytic function $D(\lambda)$ defined in a domain containing the right half of the complex λ-plane. It is proved in [1] that

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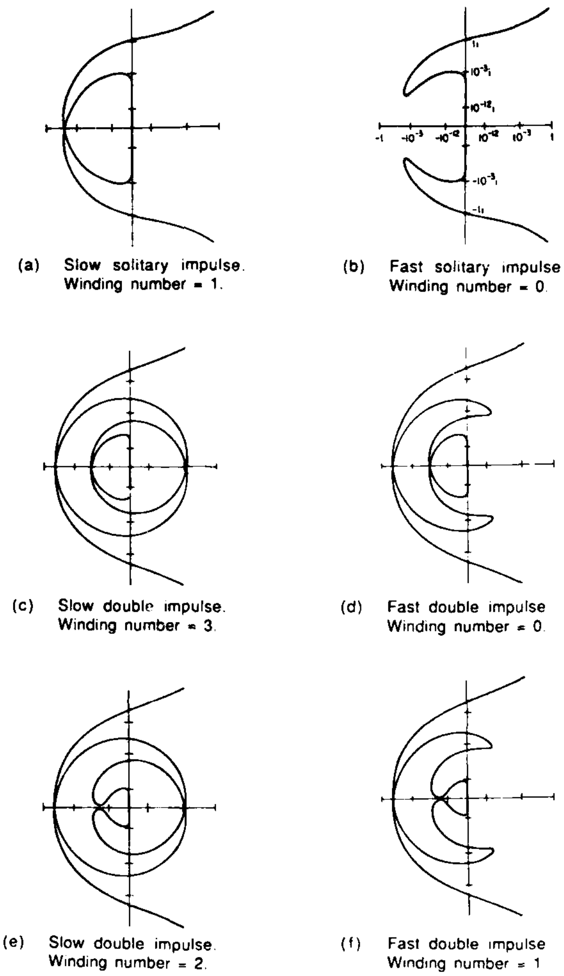

Figure 1. The image of the imaginary axis under the mapping D.

the winding number of the image of the imaginary axis under the mapping D is the same as the number of unstable modes. Since the function $D(\lambda)$ is extremely complicated, the winding number can only be found by computer. Feroe applied Evan's winding number to McKean's caricature with solitary impulses in [3] and a few double impulses in [5]. The image of the function D for n-ple impulses, with $n \geqq 3$, would be too complicated to count. However, it is interesting to look at Feroe [3] and [5]'s winding number picture, with $n \leqq 2$, for McKean's caricature (see Figure 1)[1]:

[1]Most of the figures were kindly supplied by Dr. John A. Feroe, with the permission of SIAM.

Feroe [5] also sketched the pictures of the unstable modes for double impulses and conjectured that there are only two kinds of instabilities: spacing type and amplitude type.

In this paper we give an analytical answer to both the number and the type of the unstable modes, of multiple impulse solutions consisting of any finite number of pulses, of McKean's caricature. The results are summarized in the following diagrams:

$$n \leqq \left\{ \begin{array}{l} \text{The number of unstable modes of a} \\ \text{“superposition” of } n \textit{ slow} \text{ solitary waves} \end{array} \right\} \leqq 2n-1$$

$$\swarrow \qquad \searrow$$

$$\boxed{\begin{array}{l} \text{exactly } n \text{ modes of} \\ \text{amplitude type} \end{array}} \qquad \boxed{\begin{array}{l} \text{at most } n-1 \text{ modes of} \\ \text{spacing type} \end{array}}$$

$$0 \leqq \left\{ \begin{array}{l} \text{The number of unstable modes of a} \\ \text{“superposition” of } n \textit{ fast} \text{ solitary waves} \end{array} \right\} \leqq n-1$$

$$\downarrow$$

$$\boxed{\text{they are all of spacing type}}$$

The author believes that the upper and lower bounds provided above are crucial. Proving this fact is an interesting problem for future research. As a special case, which is of most importance, when the lower bound zero of the second inequality is reached, one obtains a stable n-ple impulse. An even more important open problem is to provide criteria that determine when an n-ple impulse traveling wave is actually stable.

All of the results mentioned above are about infinitesimal stability. It would be very interesting to look at the real evolution in function space of the full nonlinear equation. McKean–Moll [13] studied this problem for a "small" function space and a special range of a, b, c.

1. The Variational Equation

We introduce the traveling coordinate frame (z, t), with $z = x + kt$, in which system (1) takes the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial z^2} - k\frac{\partial u}{\partial z} - u + H(u-a) + v, \\ \frac{\partial v}{\partial t} &= -k\frac{\partial v}{\partial z} + bu - cv. \end{aligned} \tag{2}$$

For any positive integer n, the n-ple impulse (u_k, v_k) is a t-independent solution of this system. To study its stability we consider the variational equation

$$\begin{aligned} \frac{\partial \tilde{U}}{\partial t} &= \frac{\partial^2 \tilde{U}}{\partial z^2} - k\frac{\partial \tilde{U}}{\partial z} - \tilde{U} + \delta(u_k - a)\tilde{U} + \tilde{V}, \\ \frac{\partial \tilde{V}}{\partial t} &= -k\frac{\partial \tilde{V}}{\partial z} + b\tilde{U} - c\tilde{V}. \end{aligned} \tag{3}$$

Since $u_k(0) = u_k(z_1) = \cdots = u_k(z_{2n-1}) = a$, the compound δ-function is explicitly

$$\delta(u_k - a) = \sum_{i=0}^{2n-1} \frac{\delta(z - z_i)}{|u_k'(z_i)|}. \tag{4}$$

We now look for solutions to the system (3) having the form

$$\tilde{U}(z,t) = e^{\lambda t}U(z), \qquad \tilde{V}(z,t) = e^{\lambda t}V(z). \tag{5}$$

It follows from (3) and (4) that (U', U, V) must satisfy the ordinary differential equation

$$\begin{bmatrix} U' \\ U \\ V \end{bmatrix}' = \begin{bmatrix} k & 1+\lambda & -1 \\ 1 & 0 & 0 \\ 0 & b/k & -(c+\lambda)/k \end{bmatrix} \begin{bmatrix} U' \\ U \\ V \end{bmatrix}, \tag{6}$$

for $z \neq z_i$, $i = 0, 1, 2, \cdots, 2n - 1$, and the jump conditions

$$U'\Big|_{z_i^-}^{z_i^+} = -\frac{U(z_i)}{|u_k'(z_i)|}, \qquad i = 0, 1, \cdots, 2n - 1. \tag{7}$$

The n-ple impulse u_k is unstable if the system (6) and (7) has a bounded solution with $\mathcal{Re}\,\lambda > 0$. This solution is an unstable mode with growth parameter $\mathcal{Re}\,\lambda$. On the other hand, u_k is stable if the system (6) and (7) has no bounded solutions with $\mathcal{Re}\,\lambda > 0$. For $\lambda = 0$, there is always a solution to the system (6) and (7): $(U', U, V) = (u_k'', u_k', v_k')$. This is because any translate of u_k is also a solution to the system (2).

2. The Main Results

Our rest is based on Rinzel–Keller [14]'s numerical evidence, i.e., we assume the a-k curve and a-λ curve there (see Figures 2 and 3). Their computation was carried out for the case $c = 0$, but the results remain true when c is small by continuity. The a-λ curve corresponds to the lower branch of the a-k curve, i.e.,

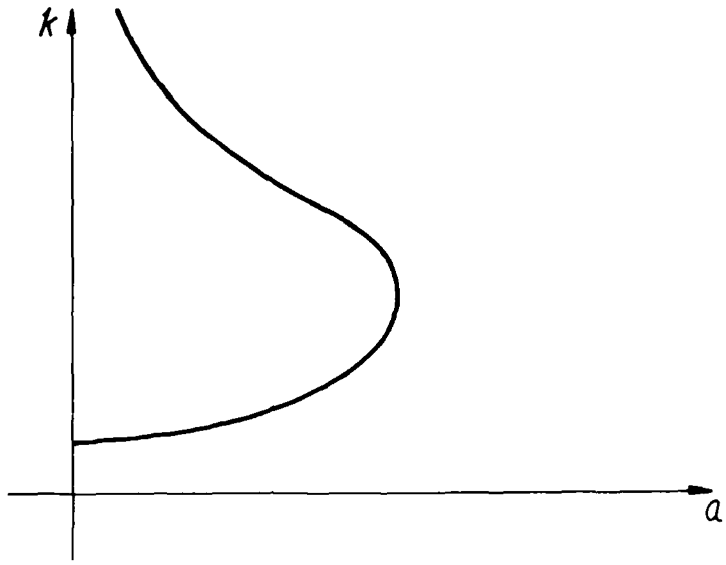

Figure 2. *a*-*k* curve, $c = 0$, $b =$ constant.

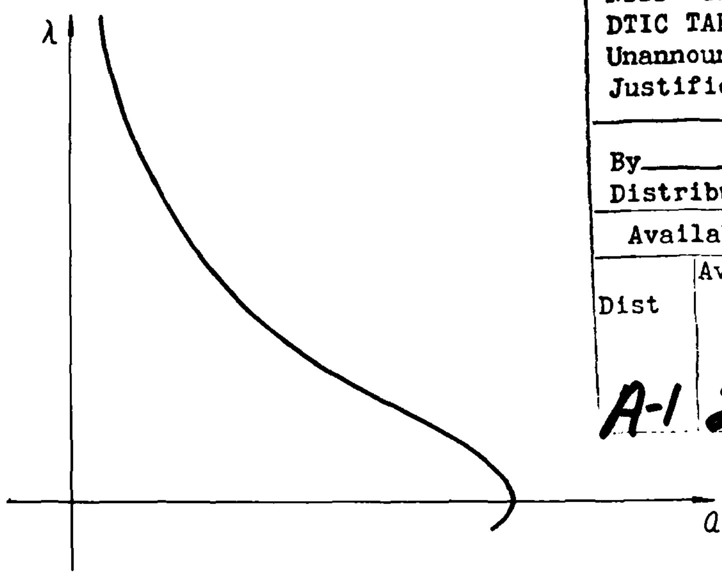

Figure 3. *a*-λ curve, $c = 0$, $b =$ constant.

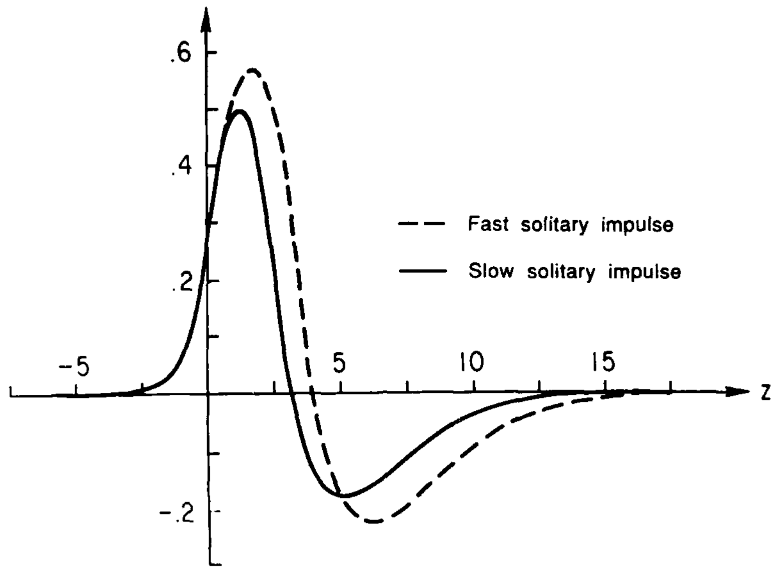

Figure 4.

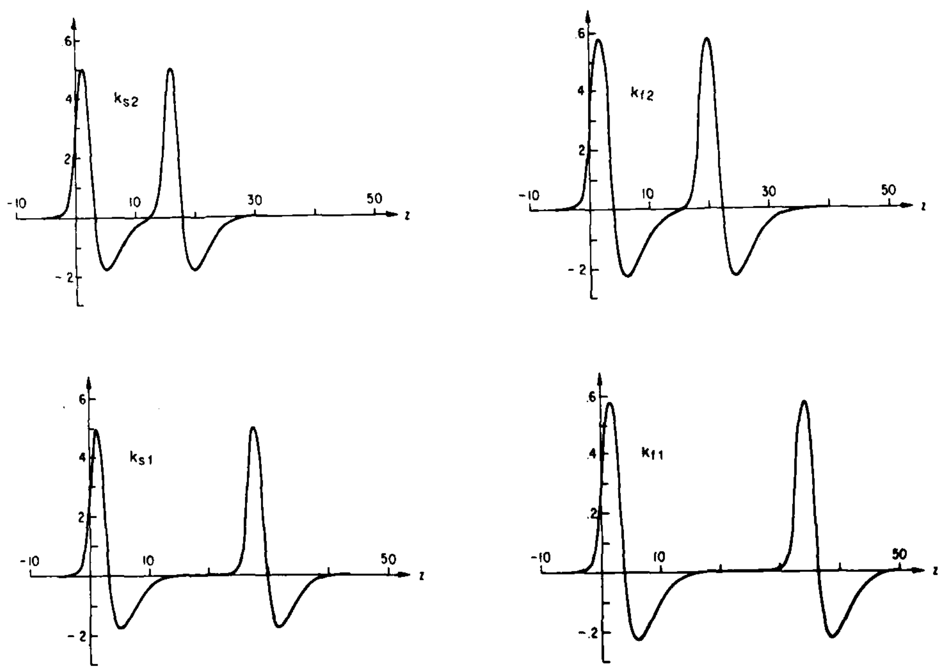

Figure 5.

if k is on the lower branch of the a-k curve, then the solitary impulse u_k has exactly one unstable mode.

THEOREM 1 (part 1). *Assume* (u_{k_0}, v_{k_0}) *is a solitary impulse solution to the system* (1) *with* k_0 *on the upper branch of the a-k curve. Then for an arbitrary positive integer n, the widely spaced n-ple impulse* (u_k, v_k), *with k close enough to* k_0, *has at most n unstable modes. The corresponding eigenvalues of the variational equation approach the origin as* $k \to k_0$.

THEOREM 1 (part 2). *Assume* k_0 *is taken on the lower branch of the a-k curve, and* λ_0 *is the corresponding eigenvalue on the a-*λ *curve. Then the widely spaced n-ple impulse* (u_k, v_k), *with k close enough to* k_0, *has at most* $2n - 1$ *and at least n unstable modes: exactly n eigenvalues of the variational equation approach* λ_0 *while the rest of them approach the origin as* $k \to k_0$.

THEOREM 2 (part 1). *The eigenvalues close to the origin, in both part* 1 *and part* 2 *of Theorem* 1, *are real and simple.*

THEOREM 2 (part 2). *The eigenvalues close to* λ_0, *in the part* 2 *of Theorem* 1 *are real and simple when a is small.*

Remark 1. By Theorems 1 and 2 we see that a widely spaced n-ple impulse solution has no complex unstable mode.

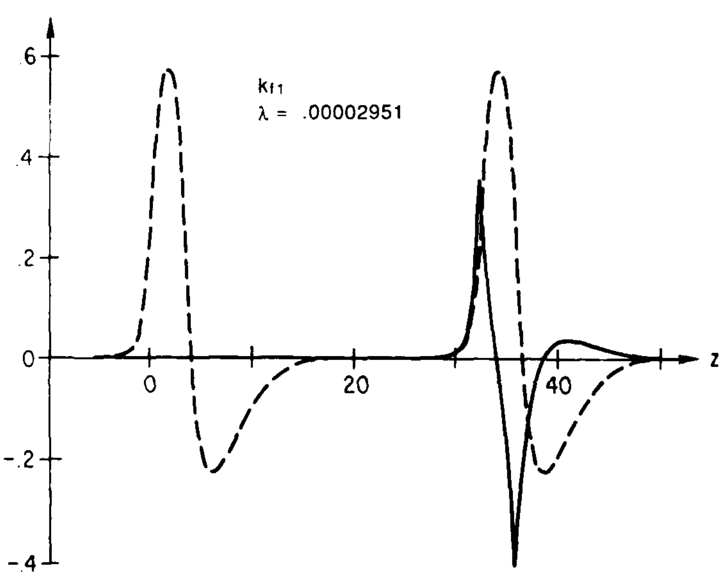

Figure 6.

Remark 2. We believe that the result of part 2 of Theorem 2 holds for all $a \in (0, a_\nu)$, where a_ν is the knee on the a-k curve. In fact, from the proof of this part, we see that this is true, at least, for a on the full interval excluding a discrete set.

Remark 3. We believe that the eigenvalues near the origin correspond to the spacings, and those near λ_0 correspond to the amplitudes. This idea is supported by Feroe [5]'s numerical computation. We recall his results as follows:

Let $a = .275$, $b = -.2$, $c = .05$, then the system (1) has exactly two solitary wave solutions: the larger one traveling with a faster velocity than the smaller. Denote the faster and slower velocities by k_f and k_s, respectively; see Figure 4.

The system (1) has countably many double impulse solutions with velocities close to either k_f or k_s. Feroe located four of them with velocities k_{f_1}, k_{f_2}, k_{s_1},

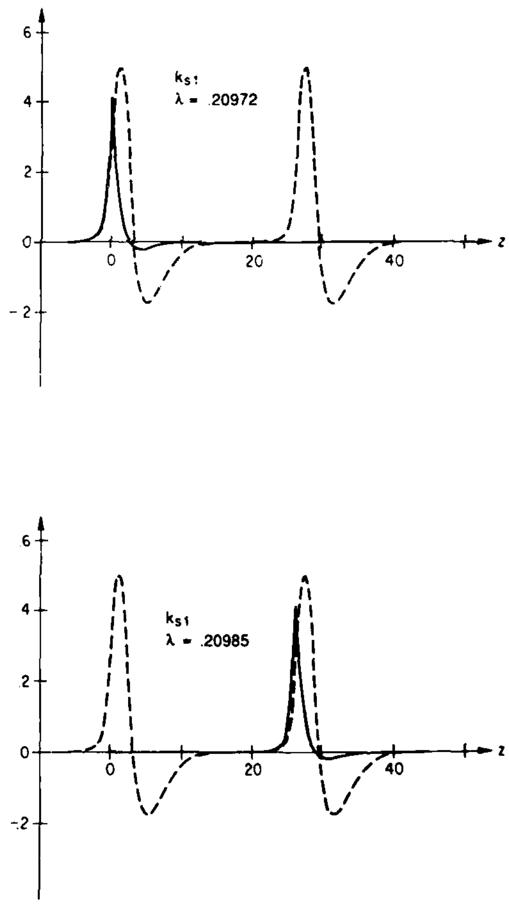

Figure 7.

and k_{s_2} satisfying $k_f < k_{f_1} < k_{f_2}$, $k_s > k_{s_1} > k_{s_2}$ and

$$k_{f_1} - k_f \approx 10^{-19}, \qquad k_{f_2} - k_f \approx 10^{-11},$$

$$k_s - k_{s_1} \approx 10^{-15}, \qquad k_s - k_{s_2} \approx 10^{-7};$$

see Figure 5.

The double impulse with velocity k_{f_2} is stable. The one with velocity k_{f_1} is unstable with only one unstable mode, the eigenvalue ≈ 0. This unstable mode is of spacing type, since it locates at one of the pulses (the second here) and raises up the front of the pulse while pulling down the back; see Figure 6. Another point of view concerning this aspect is that the unstable mode is indeed similar to

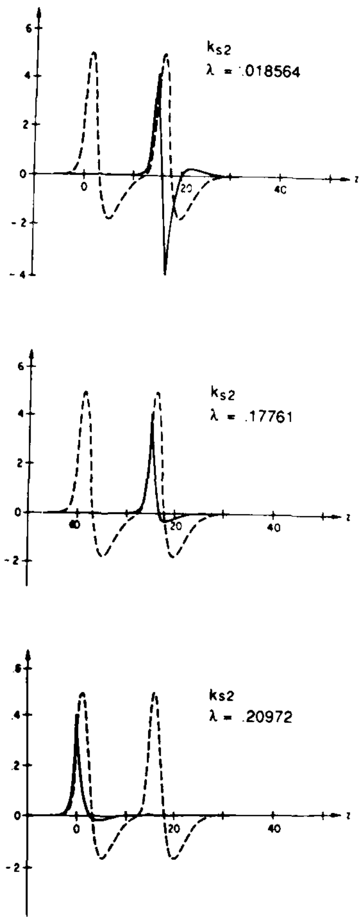

Figure 8.

the derivative of the second pulse, since the eigenvalue is approximately zero. Therefore this unstable mode causes a translation of this pulse.

The double impulse with velocity k_{s_1} has two unstable modes, which are all of amplitude type; see Figure 7.

The double impulse with velocity k_{s_2} has three unstable modes: one of spacing type and two of amplitude type; see Figure 8.

3. Construction of Unstable Modes

Let

$$A_\lambda = \begin{bmatrix} k & 1+\lambda & -1 \\ 1 & 0 & 0 \\ 0 & \frac{b}{k} & -\frac{c+\lambda}{k} \end{bmatrix}.$$

It is easy to see that A_0 is just the matrix appearing in the system (4) of [15]. The characteristic polynomial is

$$Q(\beta) = \beta^3 + \left(\frac{c+\lambda}{k} - k\right)\beta^2 - (2\lambda + c + 1)\beta + \frac{b}{k} - \frac{1}{k}(1+\lambda)(c+\lambda).$$

Define $\beta_1(\lambda)$, $\beta_2(\lambda)$, $\beta_3(\lambda)$ to be the three eigenvalues with $\beta_i(0) = \alpha_i$, $i = 1, 2, 3$. In section 10 we shall prove:

LEMMA 1. *$\mathcal{R}e\,\beta_1(\lambda) > 0$, $\mathcal{R}e\,\beta_2(\lambda) < 0$, $\mathcal{R}e\,\beta_3(\lambda) < 0$ if and only if $\lambda \in \rho^+$, where ρ^+ is the shaded region of Figure 9, with the left boundary given by*

$$\lambda = \tfrac{1}{2}\left[-(1+\theta^2+c) + \left((1+\theta^2-c)^2 + 4b\right)^{1/2}\right] - i\theta k,$$

with real parameter $-\infty < \theta < \infty$.

Solutions of the system (6) are sums of the exponentials $\exp\{\beta_i z\}Y_i$, $i = 1, 2, 3$, where the vectors Y_i's are the eigenvectors of A_λ,

$$Y_i = \begin{bmatrix} \beta_i \\ 1 \\ -\beta_i^2 + k\beta_i + 1 + \lambda \end{bmatrix}, \qquad i = 1, 2, 3. \tag{8}$$

Since β_1 is the only eigenvalue with non-negative real part, a bounded solution to

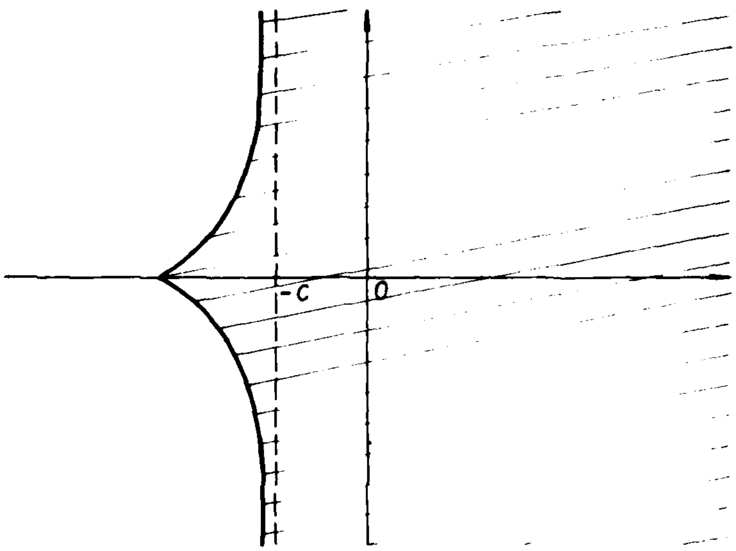

Figure 9.

(6) and (7) on the interval $(-\infty, 0)$ must be

$$\begin{bmatrix} U' \\ U \\ V \end{bmatrix} = \exp\{\beta_1 z\} \begin{bmatrix} \beta_1 \\ 1 \\ -\beta_1^2 + k\beta_1 + 1 + \lambda \end{bmatrix} \tag{9}$$

if we fix $U(0) = 1$. Let

$$\begin{bmatrix} U' \\ U \\ V \end{bmatrix} = D_j \exp\{\beta_1 z\} Y_1 + E_j \exp\{\beta_2 z\} Y_2 + F_j \exp\{\beta_3 z\} Y_3, \tag{10}$$

if $z_{j-1} < z < z_j$, $j = 1, 2, \cdots, 2n$, taking $z_{2n} = +\infty$. In view of the jump condition (7) we have

$$\begin{aligned} D_j &= 1 - \frac{k\beta_1 + \lambda + c}{kq_1'} \sum_{i=0}^{j-1} \frac{\exp\{-\beta_1 z_i\} U(z_i)}{|u_k'(z_i)|}, \\ E_j &= -\frac{k\beta_2 + \lambda + c}{kq_2'} \sum_{i=0}^{j-1} \frac{\exp\{-\beta_2 z_i\} U(z_i)}{|u_k'(z_i)|}, \\ F_j &= -\frac{k\beta_3 + \lambda + c}{kq_3'} \sum_{i=0}^{j-1} \frac{\exp\{-\beta_3 z_i\} U(z_i)}{|u_k'(z_i)|}. \end{aligned} \tag{11}$$

where $q_i' = Q'(\beta_i)$, $i = 1,2,3$. Substituting (8) and (11) into (10), we have

$$U(z) = \exp\{\beta_1 z\} - \sum_{s=1}^{3} \frac{k\beta_s + \lambda + c}{kq_s'} \sum_{i=0}^{j-1} \frac{\exp\{\beta_s(z - z_i)\}U(z_i)}{|u_k'(z_i)|},$$

if $z_{j-1} < z < z_j$, $j = 1,2,\cdots,2n$. Therefore,

$$\begin{aligned} U(z_j) &= \exp\{\beta_1 z_j\} - \sum_{s=1}^{3} \frac{k\beta_s + \lambda + c}{kq_s'} \sum_{i=0}^{j-1} \frac{\exp\{\beta_s(z_j - z_i)\}U(z_i)}{|u_k'(z_i)|} \\ &= \exp\{\beta_1 z_j\} + \sum_{i=0}^{j-1} K_\lambda(z_j - z_i)\frac{U(z_i)}{|u_k'(z_i)|}, \end{aligned} \tag{12}$$

for $j = 1,2,\cdots,2n-1$, where

$$K_\lambda(z) \stackrel{\text{def}}{=} -\sum_{s=1}^{3} \frac{k\beta_s + \lambda + c}{kq_s'} \exp\{\beta_s z\}. \tag{13}$$

Equation (12) can be written in terms of a matrix as

$$\begin{bmatrix} 1 \\ U(z_1) \\ U(z_2) \\ \vdots \\ U(z_{2n-1}) \end{bmatrix} = \tilde{M}_n(\lambda) \begin{bmatrix} 1 \\ U(z_1) \\ U(z_2) \\ \vdots \\ U(z_{2n-1}) \end{bmatrix} + \begin{bmatrix} 1 \\ \exp\{\beta_1 z_1\} \\ \exp\{\beta_1 z_2\} \\ \vdots \\ \exp\{\beta_1 z_{2n-1}\} \end{bmatrix}, \tag{14}$$

where $\tilde{M}_n(\lambda) \stackrel{\text{def}}{=} (m_{i,j})_{2n \times 2n}$ is a lower-triangular matrix, with

$$m_{i,j} = \frac{K_\lambda(z_{i-1} - z_{j-1})}{|u'(z_{j-1})|} \quad \text{if} \quad i > j,$$

$$m_{i,j} = 0 \quad \text{if} \quad i \leqq j.$$

Equation (14) implies

$$\begin{bmatrix} 1 \\ U(z_1) \\ U(z_2) \\ \vdots \\ U(z_{2n-1}) \end{bmatrix} = M_n^{-1}(\lambda) \begin{bmatrix} 1 \\ \exp\{\beta_1 z_1\} \\ \exp\{\beta_1 z_2\} \\ \vdots \\ \exp\{\beta_1 z_{2n-1}\} \end{bmatrix}, \tag{15}$$

where $M_n(\lambda) \stackrel{\text{def}}{=} I_{2n} - \tilde{M}_n(\lambda)$ and I_{2n} is the $2n \times 2n$ identity matrix.

Formula (10) combined with (11) and (15) gives the unique solution to the system (6) and (7) satisfying $U(-\infty) = 0$. Therefore, $U(z)$ is an unstable mode if and only if $D_{2n}(\lambda) = 0$.

4. The Function $T_n(\lambda, k)$

Define $T_n(\lambda, k) = D_{2n}$. By formulas (11) and (15) we see that

$$T_n(\lambda, k) = 1 + d(\lambda, k) X_{n,1} M_n^{-1} X_{n,2}',$$

where $X_{n,1}$ and $X_{n,2}$ are $2n$-dimensional vectors defined as

$$X_{n,1} = \left(\frac{1}{|u_k'(0)|}, \frac{\exp\{-\beta_1 z_1\}}{|u_k'(z_1)|}, \frac{\exp\{-\beta_1 z_2\}}{|u_k'(z_2)|}, \cdots, \frac{\exp\{-\beta_1 z_{2n-1}\}}{|u_k'(z_{2n-1})|} \right),$$

$$X_{n,2} = (1, \exp\{\beta_1 z_1\}, \exp\{\beta_1 z_2\}, \cdots, \exp\{\beta_1 z_{2n-1}\});$$

$X_{n,2}'$ denotes the transpose of $X_{n,2}$, and

$$d(\lambda, k) = -\frac{k\beta_1 + \lambda + c}{kq_1'}.$$

The function $T_n(\cdot, k)$ is a well-defined complex analytic function on the set ρ^+. This function is the principal tool in the study of the stability of the n-ple impulse solutions; λ is an eigenvalue of the variational equations (6) and (7) if and only if $T_n(\lambda, k) = 0$.

Remark. The parameter k is taken so that the z_i, $i = 1, 2, \cdots, 2n-1$, exist. Hence, the functions $T_j(\lambda, k)$, $j \leqq n$, are all well defined when k is the traveling speed of an n-ple impulse solution.

5. Four Basic Facts Concerning $T_1(\lambda, k_0)$

We summarize four basic facts concerning the function $T_1(\lambda, k_0)$ as follows.

(i) If k_0 is on the upper branch of the a-k curve, then in the half complex plane $\mathcal{R}e\,\lambda \geqq 0$, the equation $T_1(\lambda, k_0) = 0$ has an unique solution: $\lambda = 0$.

(ii) If k_0 is on the lower branch of the a-k curve, then in the half complex plane $\mathcal{R}e\,\lambda \geqq 0$, the equation $T_1(\lambda, k_0) = 0$ has two solutions: $\lambda = 0$ and $\lambda = \lambda_0$, where λ_0 is a positive number on the a-λ curve; see Figure 3.

(iii) The solution $\lambda = 0$ of both (i) and (ii) is simple.

(iv) The solution $\lambda = \lambda_0$ of (ii) is simple.

These facts were given by Rinzel–Keller [14], and Feroe [3]; (i), (ii) and (iv) were obtained numerically and (iii) was proved algebraically by Rinzel-Keller [14]. We assume these four facts in the following discussion.

6. Behavior of $T_n(\lambda, k)$ as $|\lambda| \to \infty$

LEMMA 3.

$$\lim_{|\lambda|\to\infty} T_n(\lambda, k) = 1$$

for k near k_0 and $\lambda \in \rho^+$.

Proof: The function T_n is taken as in formula (11):

$$T_n = 1 - \frac{k\beta_1 + \lambda + c}{kq_1'} \sum_{j=0}^{2n-1} \frac{\exp\{-\beta_1 z_j\} U(z_j)}{|u_k'(z_j)|}.$$

Direct computation with the aid of Lemma 2 (see Section 10) shows that

$$\frac{k\beta_1 + \lambda + c}{kq_1'} = O\left(\frac{1}{\sqrt{\lambda}}\right) \to 0.$$

Therefore it is sufficient to prove that $\exp\{-\beta_1 z_j\}U(z_j)$, $j = 1, 2, \cdots, 2n-1$, are all bounded as $|\lambda| \to \infty$ for $\lambda \in \rho^+$. This fact is seen from formula (12):

$$U(z_j) = \exp\{\beta_i z_j\} + \sum_{i=0}^{j-1} K_\lambda(z_j - z_i) \frac{U(z_i)}{|u_k'(z_i)|}.$$

Multiplying by $\exp\{-\beta_1 z_j\}$ we obtain

$$\exp\{-\beta_1 z_j\}U(z_j) = 1 + \sum_{i=0}^{j-1} \exp\{-\beta_1(z_j - z_i)\}$$
$$\times K_\lambda(z_j - z_i) \frac{\exp\{-\beta_1 z_i\}U(z_i)}{|u_k'(z_i)|},$$

in which

$$\begin{aligned}
&\exp\{-\beta_1(z_j - z_i)\}K_\lambda(z_j - z_i) \\
&\quad = -\frac{k\beta_1 + \lambda + c}{kq_1'} - \frac{k\beta_2 + \lambda + c}{kq_2'} \exp\{\beta_2(z_j - z_i) - \beta_1(z_j - z_i)\} \\
&\qquad - \frac{k\beta_3 + \lambda + c}{kq_3'} \exp\{\beta_3(z_j - z_i) - \beta_1(z_j - z_i)\} \\
&\quad = -\frac{k\beta_1 + \lambda + c}{kq_1'} + o(\exp\{-\beta_1 z_1\}) \\
&\quad = O\left(\frac{1}{\sqrt{\lambda}}\right) + o(\exp\{-\sqrt{\lambda}\, z_1\}) \\
&\quad = O\left(\frac{1}{\sqrt{\lambda}}\right)
\end{aligned}$$

as $|\lambda| \to \infty$ for $\lambda \in \rho^+$. Thus we have

$$\begin{bmatrix} 1 \\ \exp\{-\beta_1 z_1\} U(z_1) \\ \exp\{-\beta_1 z_2\} U(z_2) \\ \vdots \\ \exp\{-\beta_1 z_{2n-1}\} U(z_{2n-1}) \end{bmatrix} = \left(I_{2n} + O\left(\frac{1}{\sqrt{\lambda}}\right)\right)^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

This makes plain the fact that $\exp\{-\beta_1 z_j\} U(z_j)$, $j = 1, 2, \cdots, 2n-1$, are all bounded. The proof is finished.

7. Behavior of $T_n(\lambda, k)$ on Compacts

LEMMA 4.

$$\lim_{k \to k_0} T_n(\lambda, k) = [T_1(\lambda, k_0)]^n$$

uniformly for λ on arbitrary compact subsets of ρ^+.

Proof:

$$M_n = \begin{bmatrix} M_{n-1} & 0 \\ -N_{2\times 2(n-1)} & M_1^{(n)} \end{bmatrix},$$

where

$$M_1^{(n)} = \begin{bmatrix} 1 & 0 \\ -\dfrac{K(z_{2n-1} - z_{2n-2})}{|u_k'(z_{2n-2})|} & 1 \end{bmatrix}$$

and

$$N_{2\times 2(n-1)} = \begin{bmatrix} \dfrac{K(z_{2n-2})}{|u_k'(z_0)|} & \dfrac{K(z_{2n-2} - z_1)}{|u_k'(z_1)|} & \cdots & \dfrac{K(z_{2n-2} - z_{2n-3})}{|u_k'(z_{2n-3})|} \\ \dfrac{K(z_{2n-1})}{|u_k'(z_0)|} & \dfrac{K(z_{2n-1} - z_1)}{|u_k'(z_1)|} & \cdots & \dfrac{K(z_{2n-1} - z_{2n-3})}{|u_k'(z_{2n-3})|} \end{bmatrix}.$$

Therefore,

$$M_n^{-1} = \begin{bmatrix} M_{n-1}^{-1} & 0 \\ [M_1^{(n)}]^{-1} N_{2\times 2(n-1)} M_{n-1}^{-1} & [M_1^{(n)}]^{-1} \end{bmatrix}.$$

With this blocking, an inductive formula of the function $T_n(\lambda, k)$ is obtained as follows:

$$T_n(\lambda, k) = 1 + dX_{n,1}M_n^{-1}X_{n,2}^t$$

$$= 1 + dX_{n,1}$$

$$\cdot\begin{bmatrix} M_{n-1}^{-1}X_{n-1,2}^t \\ \left[M_1^{(n)}\right]^{-1}N_{2\times 2(n-1)}M_{n-1}^{-1}X_{n-1,2}^t + \left[M_1^{(n)}\right]^{-1}\begin{pmatrix}\exp\{\beta_1 z_{2n-2}\} \\ \exp\{\beta_1 z_{2n-1}\}\end{pmatrix}\end{bmatrix}$$

$$= 1 + dX_{n-1,1}M_{n-1}^{-1}X_{n-1,2}^t$$

$$+d\left(\frac{\exp\{-\beta_1 z_{2n-2}\}}{|u_k'(z_{2n-2})|}, \frac{\exp\{-\beta_1 z_{2n-1}\}}{|u_k'(z_{2n-1})|}\right) \tag{16}$$

$$\cdot\left[M_1^{(n)}\right]^{-1}N_{2\times 2(n-1)}M_{n-1}^{-1}X_{n-1,2}^t$$

$$+d\left(\frac{\exp\{-\beta_1 z_{2n-2}\}}{|u_k'(z_{2n-2})|}, \frac{\exp\{-\beta_1 z_{2n-1}\}}{|u_k'(z_{2n-1})|}\right)$$

$$\cdot\left[M_1^{(n)}\right]^{-1}\begin{pmatrix}\exp\{\beta_1 z_{2n-2}\} \\ \exp\{\beta_1 z_{2n-1}\}\end{pmatrix}.$$

By means of Lemma 3 of [15], we see that

$$d\left(\frac{\exp\{-\beta_1 z_{2n-2}\}}{|u_k'(z_{2n-2})|}, \frac{\exp\{-\beta_1 z_{2n-1}\}}{|u_k'(z_{2n-1})|}\right)\left[M_1^{(n)}\right]^{-1}\begin{pmatrix}\exp\{\beta_1 z_{2n-2}\} \\ \exp\{\beta_1 z_{2n-1}\}\end{pmatrix}$$

$$= d\left(\frac{1}{|u_k'(z_{2n-2})|}, \frac{\exp\{-\beta_1(z_{2n-1} - z_{2n-2})\}}{|u_k'(z_{2n-1})|}\right)$$

$$\cdot\left[M_1^{(n)}\right]^{-1}\begin{pmatrix}1 \\ \exp\{\beta_1(z_{2n-1} - z_{2n-2})\}\end{pmatrix}$$

$$= T_1(\lambda, k_0) - 1 + o(1)$$

as $k \to k_0$, uniformly for λ on compact subset of ρ^+, and that

$$d\left(\frac{\exp\{-\beta_1 z_{2n-2}\}}{|u_k'(z_{2n-2})|}, \frac{\exp\{-\beta_1 z_{2n-1}\}}{|u_k'(z_{2n-1})|}\right)[M_1^{(n)}]^{-1} N_{2\times 2(n-1)} M_{n-1}^{-1} X_{n-1,2}^t$$

$$= d\left(\frac{\exp\{-\beta_1 z_{2n-2}\}}{|u_k'(z_{2n-2})|}, \frac{\exp\{-\beta_1 z_{2n-1}\}}{|u_k'(z_{2n-1})|}\right)[M_1^{(n)}]^{-1}$$

$$\cdot\begin{bmatrix} \exp\{\beta_1 z_{2n-2}\} & 0 \\ 0 & \exp\{\beta_1 z_{2n-1}\} \end{bmatrix}$$

$$\cdot(1 + o(1))\begin{pmatrix} X_{n-1,1} \\ X_{n-1,1} \end{pmatrix} M_{n-1}^{-1} X_{n-1,2}^t$$

$$= (1 + o(1))[T_{n-1}(\lambda, k) - 1]\, d\left(\frac{\exp\{-\beta_1 z_{2n-2}\}}{|u_k'(z_{2n-2})|}, \frac{\exp\{-\beta_1 z_{2n-1}\}}{|u_k'(z_{2n-1})|}\right)$$

$$\cdot[M_1^{(n)}]^{-1}\begin{pmatrix} \exp\{\beta_1 z_{2n-2}\} \\ \exp\{\beta_1 z_{2n-1}\} \end{pmatrix}$$

$$= (1 + o(1))[T_{n-1}(\lambda, k) - 1][T_1(\lambda, k_0) - 1 + o(1)]$$

$$= T_{n-1}(\lambda, k)T_1(\lambda, k_0) - T_{n-1}(\lambda, k) - T_1(\lambda, k_0) + 1 + o(1).$$

Therefore, if we assume that

$$T_{n-1}(\lambda, k) = [T_1(\lambda, k_0)]^{n-1} + o(1),$$

then we see, by (16), that

$$T_n(\lambda, k) = T_{n-1}(\lambda, k)T_1(\lambda, k_0) + o(1)$$

$$= [T_1(\lambda, k_0)]^n + o(1).$$

The proof is finished.

8. Proof of Theorem 1

CASE 1. k_0 is taken on the upper branch of the a-k curve.

Step 1 is to take a positive number $\Lambda > 0$ so large that

$$|T_n(\lambda, k)| > \tfrac{1}{2} \quad \text{if} \quad |\lambda| \geqq \Lambda,$$

uniformly for k near k_0. This can be done by means of Lemma 3. In particular, we have

$$T_n(\lambda, k) \neq 0 \quad \text{if} \quad |\lambda| \geqq \Lambda.$$

Step 2 is to take an arbitrarily small positive number $\varepsilon > 0$, such that on the disk

$$D_\varepsilon = \{\lambda: |\lambda| \leqq \varepsilon\}$$

the function $T_1(\lambda, k_0)$ has $\lambda = 0$ as its unique zero, which is simple. This is done using fact (i) and (iii) of Section 5.

Step 3. Define

$$\Omega_{\varepsilon, \Lambda} = \{\lambda: \mathscr{Re}\,\lambda \geqq 0, \tfrac{1}{2}\varepsilon \leqq |\lambda| \leqq \Lambda\}.$$

Then $\Omega_{\varepsilon, \Lambda}$ is a compact subset of ρ^+. By Lemma 4, we have

$$T_n(\lambda, k) = [T_1(\lambda, k_0)]^n + o(1)$$

uniformly for $\lambda \in \Omega_{\varepsilon, \Lambda}$, as $k \to k_0$. It follows that

$$T_n(\lambda, k) \neq 0 \quad \text{if} \quad \lambda \in \Omega_{\varepsilon, \Lambda},$$

when k is close enough to k_0.

Step 4. By Cauchy's integral formula we have

$$\frac{\partial T_n}{\partial \lambda} = n[T_1(\lambda, k_0)]^{n-1} \frac{\partial T_1}{\partial \lambda} + o(1)$$

uniformly for λ on a small disk

$$D_\delta = \{\lambda: |\lambda| \leqq \delta\},$$

with $\delta < \varepsilon$.

Step 5. Applying Rouché's lemma to the function $T_n(\lambda, k)$ on the circle $C_\delta = \{\lambda: |\lambda| = \delta\}$, we conclude that the function $T_n(\lambda, k)$ has exactly n zeros on the disk D_δ provided k is close enough to k_0.

Now we conclude that the function $T_n(\lambda, k)$ has at most n zeros with positive real part, i.e., the n-ple impulse has at most n unstable modes in this case.

CASE 2. k_0 is taken on the lower branch of the a-k curve, and the facts (ii), (iii) and (iv) of Section 5 are assumed.

The proof is the same as that of the case 1, but with one more small disk centered at λ_0.

9. Proof of Theorem 2

For simplicity we prove this theorem for the double impulse solutions, and assume $c = 0$. The result remains true when c is small. The proof for the n-ple impulse is by induction.

Proof of part 1: As in Section 7, we define

$$M_1^{(1)} = \begin{bmatrix} 1 & 0 \\ -\dfrac{K(z_1)}{|u_k'(z_0)|} & 1 \end{bmatrix},$$

$$M_1^{(2)} = \begin{bmatrix} 1 & 0 \\ -\dfrac{K(z_3 - z_2)}{|u_k'(z_2)|} & 1 \end{bmatrix},$$

and also

$$\begin{aligned} T_1^{(1)}(\lambda, k) &= 1 + d\left(\frac{1}{|u_k'(z_0)|}, \frac{\exp\{-\beta_1 z_1\}}{|u_k'(z_1)|}\right)\left[M_1^{(1)}\right]^{-1}\begin{pmatrix} 1 \\ \exp\{\beta_1 z_1\} \end{pmatrix} \\ &= 1 + d\left[\frac{1}{|u_k'(z_0)|} + \frac{1}{|u_k'(z_1)|} + \frac{K(z_1)\exp\{-\beta_1 z_1\}}{|u_k'(z_0)|\,|u_k'(z_1)|}\right], \\ T_1^{(2)}(\lambda, k) &= 1 + d\left(\frac{1}{|u_k'(z_2)|}, \frac{\exp\{-\beta_1(z_3 - z_2)\}}{|u_k'(z_3)|}\right) \\ &\quad \cdot\left[M_1^{(2)}\right]^{-1}\begin{pmatrix} 1 \\ \exp\{\beta_1(z_3 - z_2)\} \end{pmatrix} \\ &= 1 + d\left[\frac{1}{|u_k'(z_2)|} + \frac{1}{|u_k'(z_3)|} + \frac{K(z_3 - z_2)\exp\{-\beta_1(z_3 - z_2)\}}{|u_k'(z_2)|\,|u_k'(z_3)|}\right]. \end{aligned}$$

It is clear that

$$\begin{aligned} &\lim_{k \to k_0} T_1^{(1)}(\lambda, k) = T_1(\lambda, k_0), \\ (17) \qquad &\lim_{k \to k_0} T_1^{(2)}(\lambda, k) = T_1(\lambda, k_0), \end{aligned}$$

uniformly for λ on compact subset of ρ^+. Therefore, $T_1^{(1)}(T_1^{(2)})$ has a simple real

zero near the origin, when k is close enough to k_0, since

$$\frac{\partial T_1}{\partial \lambda}(0, k_0) \neq 0$$

by fact (iii). Denote this zero by $\tilde{\lambda}_1(\tilde{\lambda}_2)$.

Step 1 is to show

$$\lim_{k \to k_0} (\tilde{\lambda}_2(k) - \tilde{\lambda}_1(k))\exp\{\gamma z_2\} = c^* \neq 0, \tag{18}$$

where c^* is a constant depends only on k_0.

Proof: Define

$$\tilde{T}_1^{(1)}(\lambda, k) = \frac{1}{d(\lambda, k)}|u_k'(z_0)||u_k'(z_1)|T_1^{(1)}(\lambda, k),$$

$$\tilde{T}_1^{(2)}(\lambda, k) = \frac{1}{d(\lambda, k)}|u_k'(z_2)||u_k'(z_3)|T_1^{(2)}(\lambda, k).$$

Then $T_1^{(1)} = 0$ ($T_1^{(2)} = 0$) if and only if $\tilde{T}_1^{(1)} = 0$ ($\tilde{T}_1^{(2)} = 0$), and

$$\lim_{k \to k_0} \frac{\partial \tilde{T}_1^{(1)}}{\partial \lambda}(0, k) = \lim_{k \to k_0} \frac{\partial \tilde{T}_1^{(2)}}{\partial \lambda}(0, k) \neq 0. \tag{19}$$

Direct computation shows

$$\begin{aligned} &\left[\tilde{T}_1^{(2)}(0, k) - \tilde{T}_1^{(1)}(0, k)\right]\exp\{\gamma z_2\} \\ &= 2\,\mathcal{Re}\left\{\frac{1}{\alpha_1}(\alpha_1 - \alpha_2)p_1' u'(z_1)\frac{\exp\{i\omega z_2\}}{p_2'}(1 - \exp\{-\alpha_2 z_1\}) \right. \\ &\qquad \left. \cdot\left[1 + \frac{\alpha_1 \exp\{-\alpha_1 z_1\}}{p_1' u'(z_1)}(\exp\{\alpha_1 z_1\} - \exp\{\alpha_2 z_1\})\right]\right\} + o(1). \end{aligned} \tag{20}$$

On the other hand, we have

$$\begin{aligned} &\exp\{\gamma z_2\} S_2(k) \\ &= 2\,\mathcal{Re}\left\{\frac{\exp\{i\omega z_2\}}{p_2'}(1 - \exp\{-\alpha_2 z_1\}) \right. \\ &\qquad \left. \cdot\left[1 + \frac{\alpha_1 \exp\{-\alpha_1 z_1\}}{u'(z_1) p_1'}(\exp\{\alpha_1 z_1\} - \exp\{\alpha_2 z_1\})\right]\right\} + o(1). \end{aligned}$$

Since we have assumed that u_k is a double impulse solution, it follows from Section 7 of [15] that

$$S_2(k) = 0.$$

Therefore,

$$\lim_{k \to k_0} \operatorname{Arg}\left\{ \frac{\exp\{i\omega z_2\}}{p_2'}(1 - \exp\{-\alpha_2 z_1\}) \right.$$
$$\left. \cdot \left[1 + \frac{\alpha_1 \exp\{-\alpha_1 z_1\}}{u'(z_1) p_1'}(\exp\{\alpha_1 z_1\} - \exp\{\alpha_2 z_1\})\right]\right\}$$
$$= \tfrac{1}{2}\pi \pmod{\pi}.$$

By (19) and (20), we see that (18) holds if and only if

$$\lim_{k \to k_0} \operatorname{Arg}\left\{ \frac{1}{\alpha_1}(\alpha_1 - \alpha_2) p_1' u'(z_1) \right\}$$
$$= \lim_{k \to k_0} \operatorname{Arg}\{(\alpha_1 - \alpha_2)\}$$
$$\neq 0 \pmod{\pi},$$

but this is the case, since

$$\alpha_1 - \alpha_2 = \alpha_1 + \gamma + i\omega,$$

with $\alpha_1 > 0$, $\gamma > 0$, and $\omega > 0$.

Remark. The computation of (20) is lengthy and we omit the details. The function S_2 is the same as in Part 1, but in a different form.

Step 2 is to refine the result of Section 7 to get

$$T_2(\lambda, k) = \tilde{T}_1^{(1)}(\lambda, k)\tilde{T}_1^{(2)}(\lambda, k) + o(\exp\{-\beta_1 z_2\}). \tag{21}$$

The proof is easy and therefore we shall not give the details.

Step 3 is to fix a small positive number δ, so that

$$\beta_1(\lambda, k) > \gamma(k) + \varepsilon \tag{22}$$

is satisfied for all $|\lambda| \leq \delta$ and k near k_0, with some suitable constant $\varepsilon > 0$. This

can be done since

$$\beta_1(\lambda, k)|_{\lambda=0} = \alpha_1(k) > \gamma(k)$$

and both α_1 and γ are continuous at $k = k_0$.

Step 4. It is seen from (17) that

$$\lim_{k \to k_0} \tilde{\lambda}_1(k) = \lim_{k \to k_0} \tilde{\lambda}_2(k) = 0.$$

In particular, we have

$$|\tilde{\lambda}_1(k)| < \tfrac{1}{2}\delta, \qquad |\tilde{\lambda}_2(k)| < \tfrac{1}{2}\delta$$

provided k is close enough to k_0. Since

$$\lim_{k \to k_0} z_2(k) = +\infty,$$

we may assume that

$$|c^*|\exp\{-\gamma z_2\} < \tfrac{1}{2}\delta.$$

As $k \to k_0$, the circles

$$C_1 = \left\{\lambda \colon |\lambda - \tilde{\lambda}_1| = \tfrac{1}{3}|c^*|\exp\{-\gamma z_2\}\right\}$$

and

$$C_2 = \left\{\lambda \colon |\lambda - \tilde{\lambda}_2| = \tfrac{1}{3}|c^*|\exp\{-\gamma z_2\}\right\}$$

are disjoint, and on these circles (22) holds. Applying Rouché's lemma to formula (21) on C_1 and C_2, respectively, we see that $T_2(\cdot, k)$ has exactly one zero in each of C_1 and C_2. Denote these zeros by λ_1 and λ_2. We claim that both λ_1 and λ_2 are real. This is because $T_2(\lambda, k)$ takes real value if λ is real, and ρ^+ is symmetric with respect to the x-axis, so that the zeros of $T_2(\cdot, h)$ are either complex conjugate pairs or real. From the proof of Theorem 1, we see that λ_1 and λ_2 are the only zeros near the origin. But they cannot be complex conjugate pairs, since both $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are real. The proof of part 1 of Theorem 2 is finished.

Proof of part 2: The formula (20) of part 1 is replaced by (20)*:

$$\begin{aligned}&\left[\tilde{T}_1^{(2)}(\lambda_0, k) - \tilde{T}_1^{(1)}(\lambda_0, k)\right]\exp\{\gamma z_2\} \\ &\quad = 2\,\mathcal{R}e\left\{\frac{\exp\{i\omega z_2\}}{p_2'}(1 - \exp\{-\alpha_2 z_1\})\Theta_1(a, k)\right\} + o(1),\end{aligned} \tag{20*}$$

and the function $\exp\{\gamma z_2\} S_2(k)$ takes the form

$$\exp\{\gamma z_2\} S_2(k) = 2\,\mathcal{R}e\left\{ \frac{\exp\{i\omega z_2\}}{p_2'} (1 - \exp\{-\alpha_2 z_1\}) \Theta_2(a, k) \right\} + o(1),$$

where

$$\begin{aligned} \Theta_1 = {} & \alpha_1 - \alpha_2 - \alpha_1 \exp\{\alpha_1 z_1\} + \alpha_2 \exp\{\alpha_2 z_1\} \\ & + \frac{u''(z_1^-)}{u'(z_1)} (\exp\{\alpha_1 z_1\} - \exp\{\alpha_2 z_1\}) \\ & - \frac{K'(z_1)\exp\{-\beta_1 z_1\} - \beta_1 K(z_1)\exp\{-\beta_1 z_1\}}{u'(z_1)} (\exp\{\alpha_1 z_1\} - \exp\{\alpha_2 z_1\}) \\ & + \frac{1}{d(\lambda, k)} \Bigg[a\alpha_1 \frac{u''(z_1^-)}{u'(z_1)} (\exp\{\alpha_1 z_1\} - \exp\{\alpha_2 z_1\}) \\ & \qquad - a\alpha_1(\alpha_1 \exp\{\alpha_1 z_1\} - \alpha_2 \exp\{\alpha_2 z_1\}) - u'(z_1)(\alpha_1 - \alpha_2) \Bigg] \end{aligned}$$

and

$$\Theta_2 = 1 + \frac{\alpha_1 \exp\{-\alpha_1 z_1\}}{u'(z_1) p_1'} (\exp\{\alpha_1 z_1\} - \exp\{\alpha_2 z_1\}).$$

If we can prove that

$$\lim_{k \to k_0} \operatorname{Arg}\{\Theta_1(a, k)\} \neq \lim_{k \to k_0} \operatorname{Arg}\{\Theta_2(a, k)\} \pmod{\pi}, \tag{23}$$

then the rest of the proof is the same as that of part 1. Let us now prove (23) for small a. Direct computation shows that

$$\operatorname*{LIM}_{a \to 0} \Theta_1(a, k) = 2\alpha_1 \left[1 - \frac{p_1'}{\alpha_1} \lim_{a \to 0} \left(a\sqrt{\lambda_0} \right) \right] (\alpha_1 - \alpha_2), \tag{24}$$

uniformly for k near k_0, where LIM denotes the set of limit points. We shall prove in Lemma 5 of Section 10 that the function $a\sqrt{\lambda_0}$ has at most two limit points as $a \to 0 : 0$ and x_0, where x_0 is a positive number depending only on parameters b and c. We shall also prove in Lemma 5 that

$$1 - \frac{p_1'}{\alpha_1} x_0 \neq 0.$$

The computation of (24) is lengthy, we omit the details. It follows from formula (24) that

$$\lim_{a\to 0} \operatorname{Arg}\{\Theta_1(a,k)\} = \operatorname{Arg}\{\alpha_1 - \alpha_2\}$$

uniformly for k near k_0. Also by direct computation we obtain

$$\lim_{a\to 0} \Theta_2(a,k) = 1 + \frac{\alpha_1}{\alpha_1^2 - p_1'}(\alpha_1 - \alpha_2).$$

It is obvious that (23) holds at this limiting case and therefore for the case of small values of a. The proof is finished.

Remark. Since both Θ_1 and Θ_2 are nice transcendental functions of a, (23) should be true at least for a on the interval $(0, a_r)$ excluding a discrete set. We believe that this discrete set is empty.

10. Proofs of Lemmas 1, 2, and 5

LEMMA 1. *Assume* $|b| < \frac{1}{4}(1-c)^2$, *then* $\mathcal{Re}\,\beta_1(\lambda) > 0$, $\mathcal{Re}\,\beta_2(\lambda) < 0$, $\mathcal{Re}\,\beta_3(\lambda) < 0$ *if* $\lambda \in \rho^+$, *where* ρ^+ *is defined as in Figure* 9.

Proof: We know that

$$\mathcal{Re}\,\beta_1(0) = \mathcal{Re}\,\alpha_1 > 0,$$

$$\mathcal{Re}\,\beta_2(0) = \mathcal{Re}\,\alpha_2 < 0,$$

$$\mathcal{Re}\,\beta_3(0) = \mathcal{Re}\,\alpha_3 < 0.$$

Therefore in a neighborhood of zero we have

$$\mathcal{Re}\,\beta_1(\lambda) > 0, \quad \mathcal{Re}\,\beta_2(\lambda) < 0, \quad \mathcal{Re}\,\beta_3(\lambda) < 0.$$

This neighborhood can be extended to a region. On the boundary of this region one of the $\beta_i(\lambda)$, $i = 1,2,3$, has zero real part, i.e., there exists a real number θ such that

$$0 = |i\theta I - A_\lambda| = \det\begin{bmatrix} i\theta - k & -(1+\lambda) & 1 \\ -1 & i\theta & 0 \\ 0 & -\dfrac{b}{k} & i\theta + \dfrac{c+\lambda}{k} \end{bmatrix}$$

$$= -\frac{1}{k}\left[(\lambda + i\theta k)^2 + (c + 1 + \theta^2)(\lambda + i\theta k) + c - b + \theta^2 c\right],$$

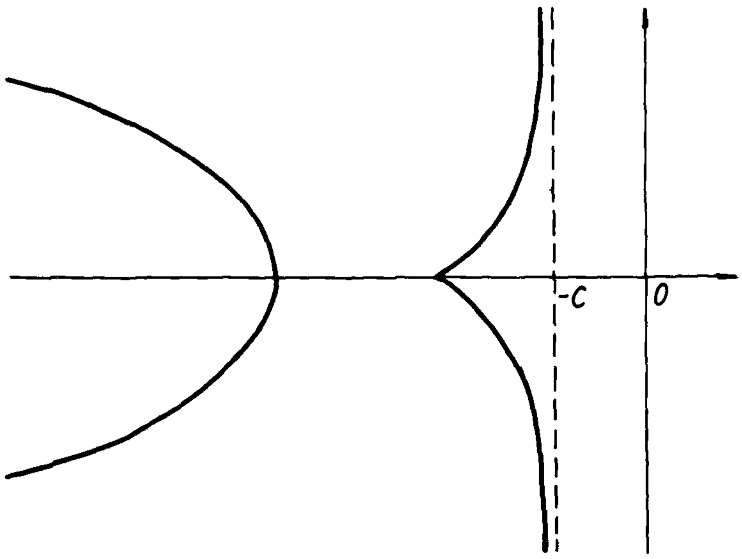

Figure 10.

which is equivalent to

$$\lambda + i\theta k = \tfrac{1}{2}\Big[-(c+1+\theta^2) \pm \big((c+1+\theta^2)^2 - 4(c-b+\theta^2 c)\big)^{1/2}\Big]$$

i.e.,

$$\lambda = \tfrac{1}{2}\Big[-(c+1+\theta^2) \pm \big((1+\theta^2-c)^2 + 4b\big)^{1/2}\Big] - i\theta k.$$

Taking θ as a real parameter we obtain two branches of continuous curves, as in Figure 10, which divide the λ-plane into three parts. In each part $\mathcal{Re}\,\beta_i(\lambda)$, $i = 1, 2, 3$, has a fixed sign, especially we have

$$\mathcal{Re}\,\beta_1(\lambda) > 0, \quad \mathcal{Re}\,\beta_2(\lambda) < 0, \quad \mathcal{Re}\,\beta_3(\lambda) < 0$$

if $\lambda \in \rho^+$.

LEMMA 2.

$$\beta_1(\lambda) = \tfrac{1}{2}\big(k + (k^2 + 4(1+\lambda))^{1/2}\big) + o(1),$$

$$\beta_2(\lambda) = \tfrac{1}{2}\big(k - (k^2 + 4(1+\lambda))^{1/2}\big) + o(1),$$

$$\beta_3(\lambda) = -\frac{\lambda}{k} - \frac{c}{k} + o(1)$$

as $|\lambda| \to \infty$ *and* $\lambda \in \rho^+$.

Proof:

$$|\beta I - A_\lambda| = \frac{1}{k}\left[\beta^2(k\beta + \lambda + c) - (k\beta + \lambda)^2 - (1 + c)(k\beta + \lambda) - (c - b)\right].$$

Let

$$f(x) = \frac{1}{k^2}(x - \lambda)^2(x + c) - x^2 - (1 + c)x - (c - b),$$

then $|\beta I - A_\lambda| = 0$ if and only if $f(k\beta + \lambda) = 0$. Direct computation shows

$$\frac{f'(x)}{f(x)} = \frac{\frac{1}{k^2}(x - \lambda)^2 + \frac{2}{k^2}(x - \lambda)(x + c) - 2x - (1 + c)}{\frac{1}{k^2}(x - \lambda)^2(x + c) - x^2 - (1 + c)x - (c - b)}.$$

Fix $|x + c| = \varepsilon$ for arbitrarily small $\varepsilon > 0$, and let $|\lambda| \to \infty$. We obtain

$$\frac{f'(x)}{f(x)} = \frac{1}{x + c} + O\left(\frac{1}{|\lambda|}\right).$$

By Rouché's lemma, we see that $f(x)$ has a zero arbitrarily close to $-c$ as $|\lambda| \to \infty$. Therefore one of the $\beta_1, \beta_2, \beta_3$ must satisfy

$$k\beta + \lambda = -c + o(1);$$

it is obvious that β_1 does not satisfy this equation if $\lambda \in \rho^+$ and $|\lambda| \to \infty$. Without loss of generality, we assume in the following discussion that

$$k\beta_3 + \lambda = -c + o(1),$$

in other words that

$$\beta_3(\lambda) = -\frac{\lambda}{k} - \frac{c}{k} + o(1).$$

To estimate β_1 and β_2, we take the characteristic polynomial in another form:

$$\begin{aligned}|\beta I - A_\lambda| &= \frac{1}{k}(k\beta + \lambda + c)\left[\beta^2 - k\beta - (1 + \lambda) + \frac{b}{k\beta + \lambda + c}\right]\\ &= \frac{1}{k}(k\beta + \lambda + c)\left[(\beta - \beta^+)(\beta - \beta^-) + \frac{b}{k\beta + \lambda + c}\right]\end{aligned}$$

with

$$\beta^{\pm} = \tfrac{1}{2}\left(k \pm (k^2 + 4(1 + \lambda))^{1/2}\right).$$

Again, by Rouché's lemma, we obtain

$$\beta_1(\lambda) = \tfrac{1}{2}\Big(k + \big(k^2 + 4(1+\lambda)\big)^{1/2}\Big) + o(1),$$

and

$$\beta_2(\lambda) = \tfrac{1}{2}\Big(k - \big(k^2 + 4(1+\lambda)\big)^{1/2}\Big) + o(1).$$

LEMMA 5. (i) *The function $a\sqrt{\lambda(a)}$ has at most two limit points as $a \to 0: 0$ and x_0, where x_0 is a positive number, and $\lambda(a)$ is the a-λ curve.*
(ii) $1 - (p_1'/\alpha_1)x_0 \neq 0$.

Proof of Lemma 5: *Step* 1. $z_1/a \to p_1'/\alpha_1$ as $a \to 0$.

Proof:

$$1 - ap_1' = \exp\{-\alpha_1 z_1\}$$

i.e.,

$$\log(1 - ap_1') = -\alpha_1 z_1$$

implies

$$ap_1' + o(a) = \alpha_1 z_1,$$

so that,

$$\frac{z_1}{a} = \frac{p_1'}{\alpha_1} + o(1).$$

Step 2. $u'(z_1)/a \to \alpha_1 - p_1'/\alpha_1$ as $a \to 0$.

Proof:

$$\begin{aligned} u'(z_1) &= a\alpha_1 \exp\{\alpha_1 z_1\} - \sum_{s=1}^{3} \frac{\alpha_s}{p_s'} \exp\{\alpha_s z_1\} \\ &= a\alpha_1 \exp\{\alpha_1 z_1\} - \sum_{s=1}^{3} \frac{\alpha_s}{p_s'} (\exp\{\alpha_s z_1\} - 1), \end{aligned}$$

since $\Sigma_{s=1}^{3}(\alpha_s/p_s') = 0$. This gives

$$\frac{u'(z_1)}{a} = \alpha_1 \exp\{\alpha_1 z_1\} - \sum_{s=1}^{3} \frac{\alpha_s^2}{p_1'} \frac{z_1}{a} \exp\{\theta_s \alpha_s z_1\}$$

$$\to \alpha_1 - \frac{p_1'}{\alpha_1} \sum_{s=1}^{3} \frac{\alpha_s^2}{p_s'}$$

$$= \alpha_1 - \frac{p_1'}{\alpha_1},$$

where θ_s, $s = 1, 2, 3$, are some intermediate values.

Step 3. Applying step 1, 2 and Lemma 2 to the equation

$$T_1(\lambda_0, k_0) = 0,$$

we have

$$1 - [1 + o(a)]\exp\left\{-2\left(a\sqrt{\lambda(a)}\right)\frac{p_1'}{\alpha_1}[1 + o(1)]\right\}$$

$$= 2\left(a\sqrt{\lambda(a)}\right)\frac{p_1'}{\alpha_1}[1 + o(1)] + 4\left(a\sqrt{\lambda(a)}\right)^2\left(\alpha_1^2 - p_1'\right)[1 + o(1)]$$

as $a \to 0$. This implies that $a\sqrt{\lambda(a)}$ is a solution to following transcendental equation (with a small perturbation):

$$1 - \exp\left\{-2\frac{p_1'}{\alpha_1}x\right\} = 2\frac{p_1'}{\alpha_1}x + 4x^2\alpha_1\left(\alpha_1 - \frac{p_1'}{\alpha_1}\right).$$

It is easy to see that this equation has at most two solutions: $x = 0$ and $x = x_0$, where x_0 is a positive number. Therefore as $a \to 0$ the function $a\sqrt{\lambda(a)}$ has at most two limit points: 0 or x_0.

Step 4 is to show that $1 - (p_1'/\alpha_1)x_0 \neq 0$.

Proof: If $x_0 = \alpha_1/p_1'$, then the transcendental equation of step 3 must be satisfied by $x = \alpha_1/p_1'$, i.e.,

$$1 - e^{-2} = 2 + 4\frac{\alpha_1^2}{(p_1')^2}\alpha_1\left(\alpha_1 - \frac{p_1'}{\alpha_1}\right),$$

so that

$$4\alpha_1^2 x_0^2 - 4\alpha_1 x_0 + (1 + e^{-2}) = 0.$$

This is impossible, since

$$``b^2 - 4ac" = -16\alpha_1^2 e^{-2} < 0.$$

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